Lecture 5 - Fundamental Theorem for Line Integrals and Green's Theorem

Math 392, section C

September 14, 2016

Last Time: *Fundamental Theorem for Line Integrals:*

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Theorem

Let C be a smooth curve, parametrized by $\vec{r}(t)$, $t \in [a, b]$.
Let f be a smooth function. Then

$$\int_C \vec{\nabla} f \circ d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

$\vec{F} = \vec{\nabla} f$

So, if a vector field is conservative, calculating line integrals is very simple.

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Question: When is $\vec{F}$ conservative?

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Question: When is $\vec{F}$ conservative?

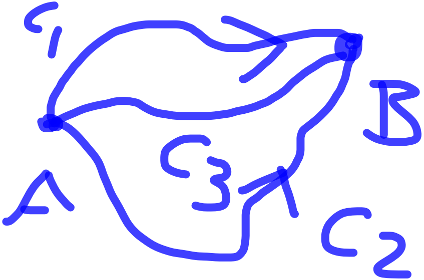
Theorem

Let $\vec{F}$ be a continuous vector field on an open, connected region D. If $\int_C \vec{F} \circ d\vec{r}$ is **independent of path**, *then $\vec{F}$ is* **conservative**.

$= \nabla f$

Definition

We say that $\int\limits_C \vec{F} \circ d\vec{r}$ is **independent of path** whenever the value of this line integral is the same for any two paths with the same endpoints.



$$\int_{C_1} F.dr = \int_{C_2} F.dr$$

For any C_1, C_2 from A to B

Definition

We say that $\int\limits_C \vec{F} \circ d\vec{r}$ is **independent of path** whenever the value of this line integral is the same for any two paths with the same endpoints.

Criterion to decide whether $\vec{F}$ is conservative

blue: 3D case

If we **knew** that

$$\vec{F} = (P, Q) = \vec{\nabla} f \qquad \vec{F} = (P, Q, R)$$

for some twice differentiable function f, then

$$\boxed{P_y = f_{xy} = f_{yx} = Q_x} \quad (*)$$

$$P_z = f_{xz} = f_{zx} = R_x$$

$$Q_z = f_{yz} = f_{zy} = R_y$$

Criterion to decide whether $\vec{F}$ is conservative

If we **knew** that

$$\vec{F} = (P, Q) = \vec{\nabla} f$$

for some twice differentiable function f, then

$$\boxed{P_y = f_{xy} = f_{yx} = Q_x}$$

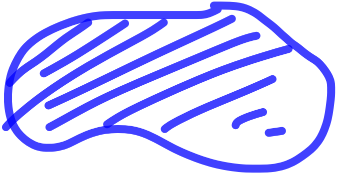
So, this is a necessary condition for a vector field to be conservative.

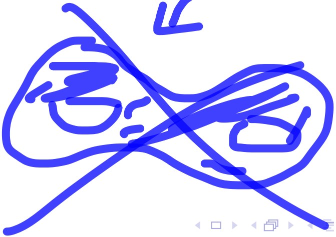
Criterion to decide whether $\vec{F}$ is conservative

Proof: Green's Thm

Theorem

If $\vec{F}(x, y) = (P(x, y), Q(x, y))$ is defined on an open, simply-connected region D, and $\boxed{P_y = Q_x}$ on D, then $\vec{F}$ is conservative.





Exercise 11 , Section 3.3:

Find the work done by $\vec{F} = (x^3y^4, x^4y^3)$ to move a particle along the curve C, parametrized by $\vec{r}(t) = (\sqrt{t}, 1 + t^3)$, $t \in [0, 1]$.

Step 0: Check if $\vec{F}$ is cons.

$P_y = 4x^3y^3$

$\quad\quad\parallel$

$Q_x = 4x^3y^3$

$\vec{F}$ is cons.!!

$\boxed{\vec{r}(0) = (0,1),\ \vec{r}(1) = (1,2)}$

Step 2: Find potential f.

$f_x = x^3 y^4$

$\int dx$

$f = \frac{x^4 y^4}{4} + g(y)$

$\frac{d}{dy}$

$f_y = x^4 y^3 + g' = x^4 y^3$

$\Rightarrow \boxed{f(x,y) = \frac{x^4 y^4}{4} + k}$

$f(1,2) - f(0,1)$

$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) = 4$

Exercise 20 , Section 3.3:

Find the work done by $\boxed{\vec{F} = (e^{-y}, -xe^{-y})}$ to move a particle from $(0, 1)$ to $(2, 0)$.

Cons? $P_y = -e^{-y}$, $Q_x = -e^{-y}$ $\Rightarrow$ $\vec{F}$ cons.

Find potential:

$f_x = e^{-y}$ $(= P)$

$$f = xe^{-y} + g(y)$$

$$f_y = -xe^{-y} + g' = -xe^{-y} \; (=Q)$$

$$\Rightarrow g' = 0 \Rightarrow g(y) = K$$

$$\Rightarrow f(x,y) = xe^{-y} + K$$

$$\int_C \vec{F} \cdot d\vec{r} = f(2,0) - f(0,1)$$

$$= \ldots$$

Exercise 31 , Section 3.3:

Let

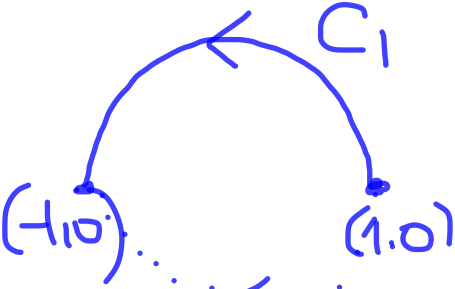
$$\vec{F} = \frac{(-y, x)}{x^2 + y^2} = ($$

Show that $\vec{F}$ satisfies $P_y = Q_x$, and that the values of the line integrals of $\vec{F}$ along the upper and lower hemispheres, joining the points $(1, 0)$ to $(-1, 0)$, are different.

$$P_y = \frac{d}{dy}\left(\frac{-y}{x^2+y^2}\right) = \frac{-1\cdot(x^2+y^2)+y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$Q_x = \frac{d}{dx}\left(\frac{x}{x^2+y^2}\right) =$$

$$\int_C \vec{F} \cdot d\vec{r}.$$



Param C_1: $\vec{r}(t) = (\cos t, \sin t)$, $t \in [0, \pi]$.

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt$$

$$= \pi$$

Param C_2:

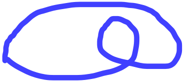
$$r(t) = (\cos(-t), \sin(-t)), \quad t \in [0,\pi]$$

$$= (\cos t, -\sin t), \quad t \in [0,\pi].$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^\pi (\sin t, \cos t) \cdot (-\sin t, -\cos t)\, dt$$

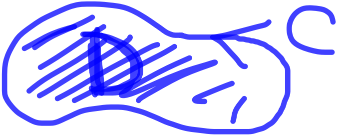
$$= \int_0^\pi -1\, dt = -\pi$$

Green's Theorem



Let C be a simple, **closed** curve on the plane, that bounds a region D.
Assume also that C is oriented **counterclockwise**[1].
If P and Q have continuous partial derivatives on an open region containing D, then

$$\oint_C P dx + Q dy = \iint_D (Q_x - P_y)\, dA$$



[1] Counterclockwise orientation will be called the *positive orientation*

Green's Theorem

Let C be a simple, **closed** curve on the plane, that bounds a region D. Assume also that C is oriented **counterclockwise**[1].
If P and Q have continuous partial derivatives on an open region containing D, then

closed

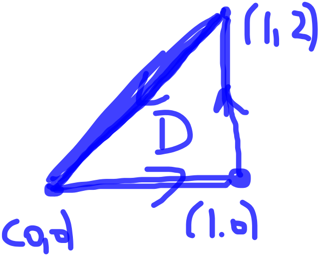
$$\boxed{\oint_C P dx + Q dy = \iint_D (Q_x - P_y)\, dA}$$

We will prove this theorem next class. Right now, we will work on how to use the theorem.

[1] Counterclockwise orientation will be called the *positive orientation*

Example: Exercise 3, Section 3.4

Evaluate $\oint_C xydx + \underline{x^2y}^3dy$, where C is the triangle with vertices $(0,0),(1,0)$ and $(1,2)$, oriented counterclockwise.

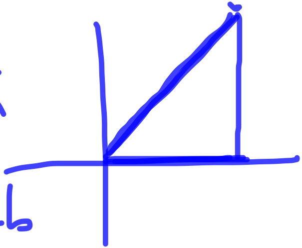


$$\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$

$$= \iint_D (2xy^3 - x)\,dA = \int_0^1 \int_0^{2x} \longrightarrow \text{PTO}$$

$D: 0 \le x \le 1, \quad 0 \le y \le 2x$

$$= \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$$



$y = mx + b$

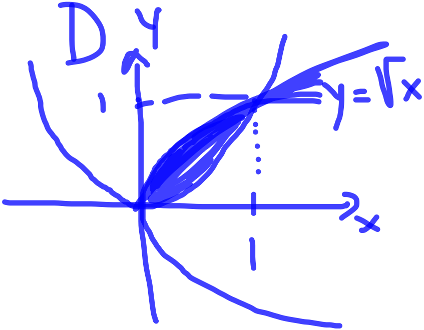
$\Rightarrow y = 2x$

Moral: See if you can use Green when asked for a line int along a simple, CLOSED curve.

Example: Exercise 9, Section 3.4

Evaluate $\oint\limits_C (y + e^{\sqrt{x}})dx + (2x + \cos(y^2))dy$, where C is the boundary of the region enclosed by $y = x^2$ and $x = y^2$, positively oriented.

$$\oint_C = \iint_D (2-1)\,dA = \iint_D dA = \int_0^1 \int_{x^2}^{\sqrt{x}} dy\,dx$$



$$0 \le x \le 1$$

$$x^2 \le y \le \sqrt{x}$$

Example: Exercise 17, Section 3.4

Find the work done by $\vec{F} = (x^2 + xy, xy^2)$ to move a particle from the origin along the x-axis to $(1, 0)$, then along a straight line to $(0, 1)$, and then back to the origin, vertically.

Example/Application

If $\vec{F}$ is such that $Q_x - P_y = 1$ (for example, if $P = -y$ and $Q = 0$, or if $Q = x$ and $P = 0$, or even if $P = -\frac{1}{2}y$ and $Q = \frac{1}{2}x$, then

$$\text{Area } (D) = \iint_D 1 \, dA$$

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$$\text{Area }(D) = \iint\limits_D 1\,dA = \oint\limits_C x\,dy$$

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$$\text{Area } (D) = \iint\limits_D 1\,dA = \oint\limits_C x\,dy = -\oint\limits_C y\,dx$$

Example/Application

If $\vec{F}$ is such that $Q_x - P_y = 1$ (for example, if $P = -y$ and $Q = 0$, or if $Q = x$ and $P = 0$, or even if $P = -\frac{1}{2}y$ and $Q = \frac{1}{2}x$, then

$$\boxed{\text{Area }(D) = \iint\limits_D 1\,dA = \oint\limits_C x\,dy = -\oint\limits_C y\,dx = \frac{1}{2}\oint\limits_C x\,dy - y\,dx}$$

Example:

Find the area of the ellipse $(bx)^2 + (ay)^2 = (ab)^2$