**Exam 3, December 3, 2015.**
**Math 308**

**Instructions:** All questions are worth the same number of points. **Important: No books, calculators, or notes are allowed. Turn off cell phones, alarms, and anything else that makes noises!** You must show **all** your work to receive credit. Any crossed out work will be disregarded (even if correct). Write **one** clear answer with a coherent derivation for each question. Good luck!

## Name:

[**1**] Prove or disprove: if $n$ is a nonnegative integer, then 5 divides $2(4^n) + 3(9^n)$.

*We proceed by induction on $n$. The base case, $n = 0$, is clear.*

*Inductive step: assume that $2(4^k) + 3(9^k)$ is divisible by 5. Then*

$$2(4^{k+1})+3(9^{k+1}) = 8(4^k)+27(9^k) = (10-2)4^k+(30-3)9^k = 5[2(4^k)+6(9^k)]-[2(4^k)+3(9^k)].$$

*The first term on the right-hand side of the equation is a multiple of 5, and by induction hypothesis, so is the second.*

*Hence, by the principle of mathematical induction, $n$ is a nonnegative integer, then 5 divides $2(4^n) + 3(9^n)$.*

[**2**] Let $A$ be a nonempty set.

**(a)** What is the definition of a relation on $A$?

*A relation on $A$ is a subset of $A \times A$.*

**(b)** Prove or disprove: The union of two equivalence relations on a nonempty set is an equivalence relation.

*Consider the set $A = \{a, b, c\}$, and the two relations*

$$R_1 = \{(a, a), (b, b), (c, c), (a, b), (b, a)\} \quad \textit{and} \quad R_2 = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}.$$

*Both are equivalence relations (reflexive, symmetric and transitive), but their union is not.*

[**3**] Prove that the multiplication in $\mathbb{Z}_n$, $n \geq 2$, defined by $[a][b] = [ab]$, is well-defined.
*We need to show that if $\alpha$ and $a$ belong to $[a]$, and $\beta$ and $b$ belong to $[b]$, then $[ab] = [\alpha\beta]$.*

*Since $\alpha$ and $a$ belong to $[a]$, and $\beta$ and $b$ belong to $[b]$, there exist $k, \ell \in \mathbb{N}$ such that $\alpha = a + kn$, and $\beta = b + \ell n$. Hence,*

$$\alpha\beta = (a + kn)(b + \ell n) = ab + n(k + \ell + nk\ell),$$

*which is congruent to $ab$ modulo $n$. Therefore, $[\alpha\beta] = [ab]$.*

**[4]** Let $A$ and $B$ be two nonempty sets such that $|A| < |B|$. Prove that $|\mathcal{P}(A)| \leq |\mathcal{P}(B)|$.

*Since $|A| < |B|$, there exists an injection $f : A \to B$. This injection induces a function $F : |\mathcal{P}(A)| \to |\mathcal{P}(B)|$: if $S \in \mathcal{P}(A)$, we define $F(S)$ by $F(S) = \{f(s); s \in S\}$.*

*We now need to show that $F$ is injective. For that, let $S, T \in \mathcal{P}(A)$, and suppose that $F(S) = F(T)$. In particular, $F(S) \subset F(T)$, that is, $\{f(s); s \in S\} \subset \{f(s); s \in T\}$. We claim that $S \subset T$.*

*Let $d \in S$. Since $F(S) \subset F(T)$, then $f(d) \in F(T)$, which implies that there exists $t \in T$ such that $f(t) = f(d)$. Since $f$ is injective, this implies that $d = t$, and so $S \subset T$.*

*Reversing the roles of $S$ and $T$, we see that $T \subset S$, hence, $S = T$. This shows that $F$ is injective.*

**[5]** Let $\mathbb{I}$ be the set of irrational numbers. Define

$$f : \mathbb{I} \to \mathbb{I} \quad \text{by} \quad f(x) = \frac{3x}{x-2}.$$

Is $f$ an injection? Is $f$ a surjection? Justify your answers.

*Note that if $y = \frac{3x}{x-2}$, then $x = \frac{2y}{y-3}$, which can be used to show that $f$ is injective.*

*Let $y \in \mathbb{I}$, and consider the number $x = \frac{2y}{y-3}$. We would like to show that $x \in \mathbb{I}$. Suppose that $x \in \mathbb{Q}$, say $x = p/q$, $p$ and $q$ integers, $q \neq 0$. We may also assume that $x \neq 2$. Then $y = \frac{3x}{x-2} = \frac{3p}{p-2q}$ would also be rational, so the map is also surjective.*