Mass rigidity of asymptotically hyperbolic spaces and some splitting theorems

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Outline

1. The mass of asymptotically hyperbolic Riemannian manifolds

2. Mass Rigidity for AH manifolds

3. Warped product splitting theorem and Obata's equation ($\text{Hess}_g f = fg$)

1. The mass of asymptotically hyperbolic Riemannian manifolds

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Asymptotically hyperbolic manifolds

1. (X. Wang 2001) Conformally compact approach
: (M^n, g) is asymptotically hyperbolic if it is conformally compact with round sphere $(\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$ at infinity and near conformal infinity the metric has an expansion

$$g = \sinh(t)^{-2}(dt^2 + g_{\mathbb{S}^{n-1}} + t^n k + O(t^{n+1})), \quad t\text{: a defining function}$$

Here, k is called the mass aspect tensor (defined on $\mathbb{S}^{n-1}$).

2. Using the chart at infinity

Definition

A Riemannian manifold (M^n, g) is **asymptotically hyperbolic** (AH) if there is a compact set $K \subset M$ and a diffeomorphism $\phi : \mathbb{R}^n \setminus B_R \to M \setminus K$ and under this diffeomorphism

$$\partial^\alpha(\phi^* g - b)_{ij} = O(r^{-q})$$

where $b = \frac{dr^2}{1+r^2} + r^2 g_{\mathbb{S}^{n-1}}, |\alpha| = 0, 1, 2$ and $q > \frac{n}{2}$.

The mass of AH manifolds

The mass of AH manifolds is defined as an $(n+1)$-dimensional vector (or the Lorentzian norm of it).

Definition (P. Chruściel and M. Herzlich '01)

The mass vector $(p_0, p_1 \ldots, p_n)$ is defined by

$$p_0 = H_g(\sqrt{1+r^2}), \quad p_i = H_g(x_i) \text{ for } i = 1, \ldots, n,$$

where

$$H_g(V) = \lim_{r\to\infty} \int_{S_r} \left[V \left(\mathring{\mathrm{div}}\, e - d(\mathring{\mathrm{tr}}\, e) \right)(\nu_0) + (\mathring{\mathrm{tr}}\, e) dV(\nu_0) - e(\mathring{\nabla} V, \nu_0) \right] d\sigma_0,$$

where $e = g - b$, ν_0 is the outward unit normal vector to $S_r = \{|x| = r\}$ and $\mathring{\mathrm{div}}, \mathring{\mathrm{tr}}, \mathring{\nabla}$, and $d\sigma_0$ are all with respect to b.

It is proved that $p_0^2 - (p_1^2 + p_2^2 + \cdots + p_n^2)$ is a geometric invariant.

Examples

1. Hyperbolic space $\to p_0 = p_1 = \cdots = p_n = 0$.

2. AdS Schwarzschild space: For $m > 0$,

$$\left((r_0, \infty) \times \mathbb{S}^{n-1}, \left(1 + r^2 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^{n-1}} \right),$$

where r_0 is the largest zero of $r^n + r^{n-2} - 2m$.

$$e = \frac{2m}{(1+r^2)(r^n + r^{n-2} - 2m)} dr^2$$

$$\to p_0 = 2(n-1)\omega_{n-1} \cdot m, \ p_1 = \cdots = p_n = 0.$$

The mass of AH manifolds

X. Wang (2001) (M, g) : conformally compact AH manifolds

$$p_0 = \int_{\mathbb{S}^{n-1}} \mathrm{tr}_{g_{\mathbb{S}^{n-1}}} \kappa \, d\sigma_{\mathbb{S}^{n-1}}, \quad p_i = \int_{\mathbb{S}^{n-1}} y_i \, \mathrm{tr}_{g_{\mathbb{S}^{n-1}}} \kappa \, d\sigma_{\mathbb{S}^{n-1}} \text{ for } i = 1, \dots, n,$$

where κ is the mass aspect tensor on $\mathbb{S}^{n-1}$ and y_i is the restriction of the Euclidean coordinates on $\mathbb{S}^{n-1}$. ($\mathrm{tr}_{g_{\mathbb{S}^{n-1}}} \kappa$ is called the **mass aspect function**.)

e.g. The metric of AdS Schwarzschild space has the expansion

$$(\sinh \rho)^{-2} \left(d\rho^2 + g_{\mathbb{S}^{n-1}} + \frac{\rho^n}{n}(2m g_{\mathbb{S}^{n-1}}) + O(\rho^{n+1}) \right),$$

thus $\kappa = 2m g_{\mathbb{S}^{n-1}}$, i.e., $\mathrm{tr}_{g_{\mathbb{S}^{n-1}}} \kappa = 2m(n-1)$. Therefore, we have

$$p_0 = 2(n-1)\omega_{n-1} \cdot m, \qquad p_1 = p_2 = \cdots = p_n = 0.$$

Positive mass theorem for AH manifolds

With this definition, one may conjecture the desired positive mass theorem as the following:

Let (M, g) be a complete AH manifold with $R_g \geq -n(n-1)$. Then the mass inequality

$$p_0^2 - (p_1^2 + \cdots + p_n^2) \geq 0$$

holds and the equality

$$p_0^2 = p_1^2 + \cdots + p_n^2$$

occurs if and only if (M^n, g) is isometric to the standard hyperbolic manifold $\mathbb{H}^n$.

Progress on the Positive mass theorem for AH manifolds

X. Wang 2001, P. Chruściel, Herzlich 2003 : Define the mass for AH manifolds and prove it is well-defined, and PMT for spin cases including rigidity.

L. Andersson, M. Cai, G. Galloway 2008 : Prove PMT for conformally compact AH manifolds ($3 \leq n \leq 7$) assuming the mass aspect function has a fixed sign including rigidity.

P. Chruściel, E. Delay 2019 : Prove PMT for conformally compact AH manifolds ($n \geq 3$) including partial rigidity when $p_0 = p_1 = \cdots = p_n = 0$. (by using gluing argument)

A. Sakovich 2020 : AH manifolds of dimension $n = 3$ with more general asymptotics, including partial rigidity when $p_0 = p_1 = \cdots = p_n = 0$ (by studying Jang equation).

Mass rigidity for AH manifolds

Theorem (Huang, J-, and Martin 2019)

Let (M, g) be a complete AH manifold without boundary. Suppose the following conditions hold:

1. $R_g \geq -n(n-1)$.
2. *The mass equality* $p_0(g) = \sqrt{p_1(g)^2 + \cdots + p_n(g)^2}$ *holds.*
3. *The mass inequality*
$$p_0^2 - (p_1^2 + \cdots + p_n^2) \geq 0$$
holds for **any AH manifold** *with* $R \geq -n(n-1)$.

Then (M, g) is isometric to the standard hyperbolic space $\mathbb{H}^n$ with sectional curvature -1.

Weighted Hölder space on AH manifolds

Consider the hyperboloid model:

$$\mathbb{H}^n \cong \left(\mathbb{R}^n, \frac{dr^2}{1+r^2} + r^2 g_{\mathbb{S}^{n-1}} \right).$$

Define

$$||f||_{C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B)} = \sum_{\ell=0,\dots,k} \sup_{\mathbb{H}^n \setminus B} r^q |\mathring{\nabla}^\ell f|_b + \sup_{\mathbb{H}^n \setminus B} r^q [\mathring{\nabla}^k f]_{\alpha;B_1(x)}.$$

Let (M,g) be an AH manifold. We define the weighted Hölder norm $||\cdot||_{C^{k,\alpha}_{-q}(M)}$ on M as the sum of $||\phi_* f||_{C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B}$ and the usual $C^{k,\alpha}$ norms on compact charts. The weighted Hölder space $C^{k,\alpha}_{-q}(M)$ is the completion of $C^{k,\alpha}_c(M)$ with respect to $||\cdot||_{C^{k,\alpha}_{-q}(M)}$.

Proof of the theorem

Let (M, g) be a AH metric satisfying the mass equality. Consider a space of AH metrics defined by

$$\mathcal{B} = \{g + h : h \in C^{2,\alpha}_{-q}(M), g + h \text{ is positive-definite}\}.$$

We will define a functional $\mathcal{F}$ on this space so that (M, g) is a minimizer of $\mathcal{F}$ subject to a scalar curvature constraint.

Let

$$f = p_0 f_0 - (p_1 f_1 + \cdots + p_n f_n)$$

be a function on M where $(p_0, p_1, \ldots, p_n)$ is the mass vector of (M, g) and f_i's are the eigenfunctions satisfying $\Delta_g f_i = n f_i$ with the asymptotics

$$f_0(x) = \sqrt{1 + r^2} + O^{2,\alpha}(r^{1-q})$$
$$f_i(x) = x_i + O^{2,\alpha}(r^{1-q}).$$

Then

$$H_g(f) = p_0 H_g(f_0) - (p_1 H_g(f_1) + \cdots + p_n H_g(f_n))$$
$$= p_0^2 - (p_1^2 + \cdots + p_n^2) = 0$$

Functional $\mathcal{F}$

Define the functional $\mathcal{F}_g$ on the space of asymptotically hyperbolic metrics

$$\begin{aligned}\mathcal{F}_g(\gamma) &= p_0(g)p_0(\gamma) - (p_1(g)p_1(\gamma) + \cdots + p_n(g)p_n(\gamma)) \\ &\qquad - \int_M (R(\gamma) + n(n-1))\, f\, d\mu_g \\ &= \int_M [L_g(\gamma - \mathfrak{b}) - (R(\gamma) + n(n-1))]\, f\, d\mu_g - \int_M (\gamma - \mathfrak{b}) \cdot L_g^* f\, d\mu_g,\end{aligned}$$

where R is the scalar curvature map, f is the function previously defined, $\mathfrak{b}$ is any fixed smooth symmetric $(0,2)$-tensor in M that coincides with the hyperbolic metric b in the chart at infinity.

$\Rightarrow$ the linearization of $\mathcal{F}_g$ can be computed as

$$D\mathcal{F}|_g(h) = -\int_M h \cdot L_g^* f\, d\mu_g.$$

Variational approach

Let $\mathcal{C}_g = \{\gamma : R(\gamma) = R_g\}$ be the constraint set. Then,

$$
\begin{aligned}
p_0(g)p_0(\gamma) - (p_1(g)p_1(\gamma) + \cdots + p_n(g)p_n(\gamma)) \\
\geq p_0(g)p_0(\gamma) - \sqrt{\sum_i p_i(g)^2}\sqrt{\sum_i p_i(\gamma)^2} \\
= p_0(g)\left(p_0(\gamma) - \sqrt{\sum_i p_i(\gamma)^2}\right) \geq 0,
\end{aligned}
$$

therefore, the metric g is a local minimizer of $\mathcal{F}_g$ in $\mathcal{C}_g$.

We can apply the method of Lagrange Multipliers **provided L_g is surjective**.

Surjectivity of L_g

The linearized scalar curvature map

$$L_g(h) = -\Delta(\mathrm{tr}_g h) + \mathrm{div}\,\mathrm{div} h - \langle h, \mathrm{Ric}_g \rangle_g$$

has been studied to solve the **scalar curvature deformation problem**
$\rightarrow$ Given (M, g) and a function f, find a tensor h so that $R_{g+h} = R_g + f$.

Fischer-Marsden '75: proved on a closed manifold
Corvino 2000: localized deformation (when f is compactly supported) ...

Theorem (Huang, J-, Martin '19)

(M^n, g) AH manifold, $k \geq 2$, and $s \in (-1, n)$. Then

$$L_g : C^{k,\alpha}_{-s}(M) \rightarrow C^{k-2,\alpha}_{-s}(M)$$

is surjective. Consequently, the scalar curvature map is locally surjective at g.

It suffices to show that $\mathrm{Im}(L_g)$ is closed and L_g^* is injective.

The mass minizing metric is static

Applying the method of Lagrange Multipliers, one can obtain $\Lambda \in (C^{0,\alpha}_{-q}(M))^*$ with

$$D\mathcal{F}|_g(h) = \langle \lambda, L_g(h) \rangle \text{ for all } h \in C^{2,\alpha}_{-q}(M).$$

Therefore we have

$$\langle \lambda, L_g(h) \rangle = -\int_M h \cdot L_g^* f \, d\mu_g \text{ for all } h \in C^{2,\alpha}_{-q}(M).$$

λ can be viewed as a distributional solution of $L_g^* \lambda = -L_g^* f$.
$\Rightarrow \Delta\lambda - n\lambda = -(\Delta f - nf) = 0.$

By using elliptic regularity and maximum principle, one can show that $\lambda = 0$.
$\Rightarrow L_g^* f = 0$, i.e., the mass minimizing metric g admits a static potential.

Definition

(M, g) is **static** if it admits a function $0 \not\equiv V \in C^2_{loc}(M)$ that satisfies $L_g^* V = 0$.
V is called a **static potential**.

Alternative mass formula with a static potential

If $V > 0$ everywhere, then there exists a solution $(\mathbb{R} \times M, h)$ of the vacuum Einstein Equations such that

$$h = -V^2\, dt^2 + g.$$

Each slice $\{t = a\} \times M$ has the same geometry, this solution can be interpreted as being "static in time."

Proposition

If V is a positive static potential, then

$$\int_M V^{-1} |\mathrm{Hess}_g V - Vg|^2 \, d\mu_g = \lim_{r \to \infty} \int_{S_r} (\mathrm{Ric} + (n-1)g)(\nabla V, \nu)\, d\sigma_g$$
$$= -\frac{n-2}{2} H_g(V).$$

In particular, $H_g(V) = 0$ implies that $\mathrm{Hess}_g V - Vg = 0$.

The equation $\mathrm{Hess}_g f - fg = 0$

So far, what we showed: (M, g) admits a positive static potential f satisfying $\mathrm{Hess}_g f - fg = 0$

(Obata '62 ,Tashiro '65, and Kanai '83) Let (M, g) be a complete Riemannian manifold (without boundary) that admits a non-trivial solution to the equation $\mathrm{Hess}_g f - fg = 0$.

1. If there exists a critical point of the solution, then the manifold is isometric to the standard hyperbolic space with curvature -1.
2. If there is no critical point, then the manifold (M^n, g) is the warped product $\mathbb{R} \times N^{n-1}$ with the metric $g = dt^2 + \xi^2 g_N$, where $\xi : \mathbb{R} \to \mathbb{R}_{>0}$ is a solution to the ODE $\xi'' - \xi = 0$. (e.g. $(\mathbb{R} \times T^{n-1}, dt^2 + e^{2t} h)$), h : flat metric on T^{n-1})

We can exclude the second case by contradiction.

The equation $\mathrm{Hess}_g f - fg = 0$ on a manifold with boundary

Question: Can we generalize Kanai's results for the case with boundary?

Proposition (Galloway and J- 2019)

$(M^n, g), n \geq 3$: noncompact, complete connected with compact connected boundary N, $f \not\equiv 0$ satisfying $\mathrm{Hess}_g f = fg$, and N : a regular level set $f^{-1}(a)$ for some $a \in \mathbb{R}_{>0}$. Then (M, g) is isometric to $[0, \infty) \times N$ equipped with the metric

$$dt^2 + \xi^2 g|_N,$$

where $\xi : [0, \infty) \to \mathbb{R}$ solves

$$\begin{cases} \xi'' - \xi = 0 \text{ on } [0, \infty), \\ \xi(0) = 1 \text{ and } \xi'(0) = \dfrac{a}{|\nabla f|_N}. \end{cases}$$

Warped product splitting theorem

Theorem (Galloway and J- '19)

$(M^n, g), n \geq 3$: complete, noncompact with compact connected boundary N. Suppose that

1. *$R_g \geq -n(n-1)$ in a neighborhood of N.*
2. *N has mean curvature $H_N \leq n-1$.*
3. *N is locally weakly outermost.*
4. *N does not carry a metric of positive scalar curvature.*
5. *$\exists$ a nontrivial solution f of $\mathrm{Hess}_g f - fg = 0$.*

Then (M, g) is isometric to $([0, \infty) \times N, dt^2 + e^{2t} g|_N)$ where $(N, g|_N)$ is Ricci flat.

Locally outermost MOTS

We say a compact boundary N to be locally weakly outermost if $\exists$ a neighborhood U of N such that there does not exist a compact hypersurface $\Sigma \subset U \setminus N$ homologous to N such that $H_\Sigma < n-1$.
$\Leftrightarrow$ N is locally weakly outermost in the initial data set $(M, g, -g)$.

$H_N \leq n-1$ and N being locally weakly outermost implies that $H_N = n-1$ by perturbation argument, i.e., N is a marginally outer trapped surface (MOTS).

Rigidity of MOTS

Theorem (Galloway-Schoen '06, Galloway '16)

(M, g) satisfying $R_g \geq -n(n-1)$. Suppose Σ is a locally weakly outermost MOTS in M that does not admit a metric of positive scalar curvature. Then

1. *Σ is Ricci flat.*
2. $g|_{T\Sigma} = A$ **and** $R_g = -n(n-1)$ *along Σ.*
3. *$\exists$ an outer neighborhood $U \cong [0, \delta) \times \Sigma$ of Σ in M such that each slice $\Sigma_t = \{t\} \times \Sigma, t \in [0, \delta)$ has* **the mean curvature** $H_{\Sigma_t} = n - 1$.

Local version

The following can be obtained by applying the previous theorem.

Theorem (Local version, Galloway and J- '19)

$(M^n, g), n \geq 3$ with compact connected boundary N. Suppose that

1. *$R_g \geq -n(n-1)$ in a neighborhood of N.*
2. *N has mean curvature $H_N \leq n-1$.*
3. *N is locally weakly outermost.*
4. *N does not carry a metric of positive scalar curvature.*

Then $\exists$ a nbd U of N that $(U, g|_U)$ is isometric to $([0,\infty) \times N, dt^2 + e^{2t} g|_N)$ where $(N, g|_N)$ is Ricci flat.

Conditions 3 and 4 are needed

The following examples are not isometric to the resulting warped product.

1. Toroidal Kottler metric with $m > 0$: $[r_0, \infty) \times T^{n-1}$ with the metric

$$g = \left(r^2 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 h$$

where $r_0 = (2m)^{\frac{1}{n}}$ and h is a flat metric on T^{n-1}.
$\Rightarrow R_g = -n(n-1)$ and $N \cong T^{n-1}$ does not carry a metric of positive scalar curvature. But N **is not weakly outermost**.

2. AdS Schwarzschild manifold: for $m > 0$, $[r_m, \infty) \times \mathbb{S}^{n-1}$ with the metric

$$g = \left(1 + r^2 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^{n-1}}, \quad r_m = (2m)^{\frac{1}{n-2}}.$$

$\Rightarrow R_g = -n(n-1)$ and N is weakly outermost. But N **carries a metric of positive scalar curvature**.

Warped product splitting theorem

Theorem (Galloway and J- '19)

$(M^n, g), n \geq 3$: complete, noncompact with compact connected boundary N. Suppose that

1. *$R_g \geq -n(n-1)$ in a neighborhood of N.*
2. *N has mean curvature $H_N \leq n-1$.*
3. *N is locally weakly outermost.*
4. *N does not carry a metric of positive scalar curvature.*
5. *$\exists$ a nonzero function f satisfying $Hess_g f - fg = 0$.*

Then (M, g) is isometric to $([0, \infty) \times N, dt^2 + e^{2t} g|_N)$ where $(N, g|_N)$ is Ricci flat.

Using the local version, one can prove such function f must be constant on N.
$\Rightarrow$ The theorem follows by the proposition.

Remark

One can obtain a similar characterization of the shrinking end by using (M, g, g).

Theorem (Galloway and J- '19)

$(M^n, g), n \geq 3$: complete, noncompact with compact connected boundary N. Suppose that

1. *$R_g \geq -n(n-1)$ in a neighborhood of N.*
2. *N has mean curvature $H_N \leq -(n-1)$.*
3. *N is locally weakly outermost (the inequality changes to $H_\Sigma < -(n-1)$).*
4. *N does not carry a metric of positive scalar curvature.*
5. *$\exists$ a nontrivial solution f of $\mathrm{Hess}_g f - fg = 0$.*

Then (M, g) is isometric to $([0, \infty) \times N, dt^2 + e^{-2t} g|_N)$ where $(N, g|_N)$ is Ricci flat.

Thank you for your attention.