Mass rigidity of asymptotically hyperbolic spaces and some splitting theorems

Hyun Chul Jang

University of Connecticut

CUNY Geometric Analysis Seminar

July 16, 2020
Outline

1. The mass of asymptotically hyperbolic Riemannian manifolds

2. Mass Rigidity for AH manifolds

3. Warped product splitting theorem and Obata’s equation ($\text{Hess}_g f = fg$)
1. The mass of asymptotically hyperbolic Riemannian manifolds

2. Mass Rigidity for AH manifolds

3. Warped product splitting theorem and Obata’s equation ($\text{Hess}_g f = fg$)
Asymptotically hyperbolic manifolds

1. (X. Wang 2001) Conformally compact approach

\((M^n, g)\) is asymptotically hyperbolic if it is conformally compact with round sphere \((\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})\) at infinity and near conformal infinity the metric has an expansion

\[
g = \sinh(t)^{-2}(dt^2 + g_{\mathbb{S}^{n-1}} + t^n k + O(t^{n+1})), \quad t: \text{a defining function}
\]

Here, \(k\) is called the mass aspect tensor (defined on \(\mathbb{S}^{n-1}\)).

2. Using the chart at infinity

**Definition**

A Riemannian manifold \((M^n, g)\) is **asymptotically hyperbolic** (AH) if there is a compact set \(K \subset M\) and a diffeomorphism \(\phi : \mathbb{R}^n \setminus B_R \rightarrow M \setminus K\) and under this diffeomorphism

\[
\partial^\alpha (\phi^* g - b)_{ij} = O(r^{-q})
\]

where \(b = \frac{dr^2}{1+r^2} + r^2 g_{\mathbb{S}^{n-1}}, |\alpha| = 0, 1, 2\) and \(q > \frac{n}{2}\).
The mass of AH manifolds

The mass of AH manifolds is defined as an \((n + 1)\)-dimensional vector (or the Lorentzian norm of it).

**Definition (P. Chruściel and M. Herzlich ’01)**

The mass vector \((p_0, p_1 \ldots, p_n)\) is defined by

\[
p_0 = H_g(\sqrt{1 + r^2}), \quad p_i = H_g(x_i) \text{ for } i = 1, \ldots, n,
\]

where

\[
H_g(V) = \lim_{r \to \infty} \int_{S_r} \left[ V \left( \text{d} \text{iv } e - d(\text{tr } e) \right) (\nu_0) + (\text{tr } e) dV(\nu_0) - e(\nabla V, \nu_0) \right] d\sigma_0,
\]

where \(e = g - b\), \(\nu_0\) is the outward unit normal vector to \(S_r = \{|x| = r\}\) and \(\text{d} \text{iv}, \text{tr}, \nabla\), and \(d\sigma_0\) are all with respect to \(b\).

It is proved that \(p_0^2 - (p_1^2 + p_2^2 + \cdots + p_n^2)\) is a geometric invariant.
Examples

1. Hyperbolic space \( p_0 = p_1 = \cdots = p_n = 0 \).

2. AdS Schwarzschild space: For \( m > 0 \),

\[
\left( (r_0, \infty) \times S^{n-1}, \left( 1 + r^2 - \frac{2m}{r^{n-2}} \right)^{-1} dr^2 + r^2 g_{S^{n-1}} \right),
\]

where \( r_0 \) is the largest zero of \( r^n + r^{n-2} - 2m \).

\[
e = \frac{2m}{(1 + r^2)(r^n + r^{n-2} - 2m)} dr^2
\]

\( \rightarrow p_0 = 2(n-1)\omega_{n-1} \cdot m, \ p_1 = \cdots = p_n = 0 \).
The mass of AH manifolds

X. Wang (2001) \((M, g)\): conformally compact AH manifolds

\[
    p_0 = \int_{S^{n-1}} \text{tr} g_{S^{n-1}} \kappa \, d\sigma_{S^{n-1}}, \quad p_i = \int_{S^{n-1}} y_i \text{tr} g_{S^{n-1}} \kappa \, d\sigma_{S^{n-1}} \text{ for } i = 1, \ldots, n,
\]

where \(\kappa\) is the mass aspect tensor on \(S^{n-1}\) and \(y_i\) is the restriction of the Euclidean coordinates on \(S^{n-1}\). (\(\text{tr} g_{S^{n-1}} \kappa\) is called the mass aspect function.)

e.g. The metric of AdS Schwarzschild space has the expansion

\[
    (\sinh \rho)^{-2} \left( d\rho^2 + g_{S^{n-1}} + \frac{\rho^n}{n} (2mg_{S^{n-1}}) + O(\rho^{n+1}) \right),
\]

thus \(\kappa = 2mg_{S^{n-1}},\) i.e., \(\text{tr} g_{S^{n-1}} \kappa = 2m(n-1)\). Therefore, we have

\[
    p_0 = 2(n-1)\omega_{n-1} \cdot m, \quad p_1 = p_2 = \cdots = p_n = 0.
\]
Positive mass theorem for AH manifolds

With this definition, one may conjecture the desired positive mass theorem as the following:

Let $(M, g)$ be a complete AH manifold with $R_g \geq -n(n - 1)$. Then the mass inequality

$$p_0^2 - (p_1^2 + \cdots + p_n^2) \geq 0$$

holds and the equality

$$p_0^2 = p_1^2 + \cdots + p_n^2$$

occurs if and only if $(M^n, g)$ is isometric to the standard hyperbolic manifold $\mathbb{H}^n$. 
Progress on the Positive mass theorem for AH manifolds

X. Wang 2001, P. Chruściel, Herzlich 2003: Define the mass for AH manifolds and prove it is well-defined, and PMT for spin cases including rigidity.

L. Andersson, M. Cai, G. Galloway 2008: Prove PMT for conformally compact AH manifolds ($3 \leq n \leq 7$) assuming the mass aspect function has a fixed sign including rigidity.

P. Chruściel, E. Delay 2019: Prove PMT for conformally compact AH manifolds ($n \geq 3$) including partial rigidity when $p_0 = p_1 = \cdots = p_n = 0$. (by using gluing argument)

A. Sakovich 2020: AH manifolds of dimension $n = 3$ with more general asymptotics, including partial rigidity when $p_0 = p_1 = \cdots = p_n = 0$ (by studying Jang equation).
The mass of asymptotically hyperbolic Riemannian manifolds

Mass Rigidity for AH manifolds

Warped product splitting theorem and Obata’s equation ($\text{Hess}_g f = fg$)
Mass rigidity for AH manifolds

Theorem (Huang, J-, and Martin 2019)

Let \((M, g)\) be a complete AH manifold without boundary. Suppose the following conditions hold:

1. \(R_g \geq -n(n-1)\).
2. The mass equality \(p_0(g) = \sqrt{p_1(g)^2 + \cdots + p_n(g)^2}\) holds.
3. The mass inequality 
   \[p_0^2 - (p_1^2 + \cdots + p_n^2) \geq 0\]
   holds for any AH manifold with \(R \geq -n(n-1)\).

Then \((M, g)\) is isometric to the standard hyperbolic space \(\mathbb{H}^n\) with sectional curvature \(-1\).
Weighted Hölder space on AH manifolds

Consider the hyperboloid model:

\[
\mathbb{H}^n \cong \left( \mathbb{R}^n, \frac{dr^2}{1 + r^2} + r^2 g_{\mathbb{S}^{n-1}} \right).
\]

Define

\[
\|f\|_{C_{-q}^{k,\alpha}(\mathbb{H}^n \setminus B)} = \sum_{\ell=0,\ldots,k} \sup_{\mathbb{H}^n \setminus B} r^q |\nabla^\ell f|_b + \sup_{\mathbb{H}^n \setminus B} r^q |\nabla^k f|_{\alpha;B_1(x)}.
\]

Let \((M, g)\) be an AH manifold. We define the weighted Hölder norm \(\| \cdot \|_{C_{-q}^{k,\alpha}(M)}\) on \(M\) as the sum of \(\|\phi_*f\|_{C_{-q}^{k,\alpha}(\mathbb{H}^n \setminus B)}\) and the usual \(C^{k,\alpha}\) norms on compact charts. The weighted Hölder space \(C_{-q}^{k,\alpha}(M)\) is the completion of \(C_c^{k,\alpha}(M)\) with respect to \(\| \cdot \|_{C_{-q}^{k,\alpha}(M)}\).
Proof of the theorem
Let \((M, g)\) be a AH metric satisfying the mass equality. Consider a space of AH metrics defined by

\[ \mathcal{B} = \{ g + h : h \in C^{2,\alpha}_-(M), g + h \text{ is positive-definite} \}. \]

We will define a functional \(F\) on this space so that \((M, g)\) is a minimizer of \(F\) subject to a scalar curvature constraint.

Let

\[ f = p_0 f_0 - (p_1 f_1 + \cdots + p_n f_n) \]

be a function on \(M\) where \((p_0, p_1, \ldots, p_n)\) is the mass vector of \((M, g)\) and \(f_i\)'s are the eigenfunctions satisfying \(\Delta_g f_i = nf_i\) with the asymptotics

\[ f_0(x) = \sqrt{1 + r^2} + O^{2,\alpha}(r^{1-q}) \]
\[ f_i(x) = x_i + O^{2,\alpha}(r^{1-q}). \]

Then

\[ H_g(f) = p_0 H_g(f_0) - (p_1 H_g(f_1) + \cdots + p_n H_g(f_n)) \]
\[ = p_0^2 - (p_1^2 + \cdots + p_n^2) = 0 \]
Define the functional $\mathcal{F}_g$ on the space of asymptotically hyperbolic metrics

$$\mathcal{F}_g(\gamma) = p_0(g)p_0(\gamma) - (p_1(g)p_1(\gamma) + \cdots + p_n(g)p_n(\gamma))$$

$$- \int_M (R(\gamma) + n(n-1)) f \, d\mu_g$$

$$= \int_M [L_g(\gamma - b) - (R(\gamma) + n(n-1))] f \, d\mu_g - \int_M (\gamma - b) \cdot L_g^* f \, d\mu_g,$$

where $R$ is the scalar curvature map, $f$ is the function previously defined, $b$ is any fixed smooth symmetric $(0, 2)$-tensor in $M$ that coincides with the hyperbolic metric $b$ in the chart at infinity.

$\Rightarrow$ the linearization of $\mathcal{F}_g$ can be computed as

$$D\mathcal{F}|_g(h) = - \int_M h \cdot L_g^* f \, d\mu_g.$$
Variational approach

Let $C_g = \{ \gamma : R(\gamma) = R_g \}$ be the constraint set. Then,

$$p_0(g)p_0(\gamma) - (p_1(g)p_1(\gamma) + \cdots + p_n(g)p_n(\gamma))$$

$$\geq p_0(g)p_0(\gamma) - \sqrt{\sum_i p_i(g)^2} \sqrt{\sum_i p_i(\gamma)^2}$$

$$= p_0(g) \left( p_0(\gamma) - \sqrt{\sum_i p_i(\gamma)^2} \right) \geq 0,$$

therefore, the metric $g$ is a local minimizer of $F_g$ in $C_g$.

We can apply the method of Lagrange Multipliers provided $L_g$ is surjective.
**Surjectivity of** $L_g$

The linearized scalar curvature map

$$L_g(h) = -\Delta(\text{tr}_g h) + \text{div div}h - \langle h, \text{Ric}_g \rangle_g$$

has been studied to solve the **scalar curvature deformation problem**

→ Given $(M, g)$ and a function $f$, find a tensor $h$ so that $R_{g+h} = R_g + f$.

Fischer-Marsden ’75: proved on a closed manifold
Corvino 2000: localized deformation (when $f$ is compactly supported) ...

**Theorem (Huang, J-, Martin ’19)**

$(M^n, g)$ AH manifold, $k \geq 2$, and $s \in (-1, n)$. Then

$$L_g : C^{k,\alpha}_{-s}(M) \to C^{k-2,\alpha}_{-s}(M)$$

is surjective. Consequently, the scalar curvature map is locally surjective at $g$.

It suffices to show that $\text{Im}(L_g)$ is closed and $L_g^*$ is injective.
The mass minimizing metric is static

Applying the method of Lagrange Multipliers, one can obtain $\Lambda \in (C_{-q}^{0,\alpha}(M))^*$ with

$$D\mathcal{F}|_{g}(h) = \langle \lambda, L_{g}(h) \rangle \text{ for all } h \in C_{-q}^{2,\alpha}(M).$$

Therefore we have

$$\langle \lambda, L_{g}(h) \rangle = -\int_{M} h \cdot L_{g}^{*}f \, d\mu_{g} \text{ for all } h \in C_{-q}^{2,\alpha}(M).$$

$\lambda$ can be viewed as a distributional solution of $L_{g}^{*}\lambda = -L_{g}^{*}f$.

$$\Rightarrow \Delta \lambda - n\lambda = -(\Delta f - nf) = 0.$$

By using elliptic regularity and maximum principle, one can show that $\lambda = 0$.

$$\Rightarrow L_{g}^{*}f = 0, \text{ i.e., the mass minimizing metric } g \text{ admits a static potential.}$$

**Definition**

$(M, g)$ is **static** if it admits a function $0 \neq V \in C_{loc}^{2}(M)$ that satisfies $L_{g}^{*}V = 0$. $V$ is called a **static potential**.
Alternative mass formula with a static potential

If $V > 0$ everywhere, then there exists a solution $(\mathbb{R} \times M, h)$ of the vacuum Einstein Equations such that

$$h = -V^2 dt^2 + g.$$ 

Each slice $\{t = a\} \times M$ has the same geometry, this solution can be interpreted as being “static in time.”

**Proposition**

If $V$ is a positive static potential, then

$$\int_M V^{-1} |\text{Hess}_g V - V g|^2 \, d\mu_g = \lim_{r \to \infty} \int_{S_r} (\text{Ric} + (n - 1)g)(\nabla V, \nu) \, d\sigma_g$$

$$= -\frac{n - 2}{2} H_g(V).$$

In particular, $H_g(V) = 0$ implies that $\text{Hess}_g V - V g = 0.$
The equation $\text{Hess}_g f - fg = 0$

So far, what we showed: $(M, g)$ admits a positive static potential $f$ satisfying $\text{Hess}_g f - fg = 0$

(Obata '62, Tashiro '65, and Kanai '83) Let $(M, g)$ be a complete Riemannian manifold (without boundary) that admits a non-trivial solution to the equation $\text{Hess}_g f - fg = 0$.

1. If there exists a critical point of the solution, then the manifold is isometric to the standard hyperbolic space with curvature $-1$.

2. If there is no critical point, then the manifold $(M^n, g)$ is the warped product $\mathbb{R} \times N^{n-1}$ with the metric $g = dt^2 + \xi^2 g_N$, where $\xi: \mathbb{R} \to \mathbb{R}_{>0}$ is a solution to the ODE $\xi'' - \xi = 0$. (e.g. $(\mathbb{R} \times T^{n-1}, dt^2 + e^{2t} h)$, $h$: flat metric on $T^{n-1}$)

We can exclude the second case by contradiction.
1. The mass of asymptotically hyperbolic Riemannian manifolds

2. Mass Rigidity for AH manifolds

3. Warped product splitting theorem and Obata’s equation ($\text{Hess}_g f = f g$)
The equation $\text{Hess}_g f - fg = 0$ on a manifold with boundary

Question: Can we generalize Kanai’s results for the case with boundary?

**Proposition (Galloway and J- 2019)**

$(M^n, g), n \geq 3$: noncompact, complete connected with compact connected boundary $N$, $f \not\equiv 0$ satisfying $\text{Hess}_g f = fg$, and $N$: a regular level set $f^{-1}(a)$ for some $a \in \mathbb{R}_{>0}$. Then $(M, g)$ is isometric to $[0, \infty) \times N$ equipped with the metric

$$dt^2 + \xi^2 g|_N,$$

where $\xi: [0, \infty) \to \mathbb{R}$ solves

$$\begin{cases} 
\xi'' - \xi = 0 \text{ on } [0, \infty), \\
\xi(0) = 1 \text{ and } \xi'(0) = \frac{a}{|\nabla f|_N}. 
\end{cases}$$
Warped product splitting theorem

Theorem (Galloway and J- '19)

\((M^n, g), n \geq 3\) : complete, noncompact with compact connected boundary \(N\).
Suppose that

1. \(R_g \geq -n(n-1)\) in a neighborhood of \(N\).
2. \(N\) has mean curvature \(H_N \leq n-1\).
3. \(N\) is locally weakly outermost.
4. \(N\) does not carry a metric of positive scalar curvature.
5. \(\exists\) a nontrivial solution \(f\) of \(\text{Hess}_g f - fg = 0\).

Then \((M, g)\) is isometric to \([0, \infty) \times N, dt^2 + e^{2t} g|_N\) where \((N, g|_N)\) is Ricci flat.
Locally outermost MOTS

We say a compact boundary $N$ to be locally weakly outermost if $\exists$ a neighborhood $U$ of $N$ such that there does not exist a compact hypersurface $\Sigma \subset U \setminus N$ homologous to $N$ such that $H_{\Sigma} < n - 1$.

$\iff$ $N$ is locally weakly outermost in the initial data set $(M, g, -g)$.

$H_N \leq n - 1$ and $N$ being locally weakly outermost implies that $H_N = n - 1$ by perturbation argument, i.e., $N$ is a marginally outer trapped surface (MOTS).
Rigidity of MOTS

Theorem (Galloway-Schoen ’06, Galloway ’16)

\((M, g)\) satisfying \(R_g \geq -n(n-1)\). Suppose \(\Sigma\) is a locally weakly outermost MOTS in \(M\) that does not admit a metric of positive scalar curvature. Then

1. \(\Sigma\) is Ricci flat.
2. \(g|_{T\Sigma} = A\) and \(R_g = -n(n-1)\) along \(\Sigma\).
3. \(\exists\) an outer neighborhood \(U \cong [0, \delta) \times \Sigma\) of \(\Sigma\) in \(M\) such that each slice \(\Sigma_t = \{t\} \times \Sigma, t \in [0, \delta)\) has the mean curvature \(H_{\Sigma_t} = n - 1\).
The following can be obtained by applying the previous theorem.

**Theorem (Local version, Galloway and J- ’19)**

\((M^n, g), n \geq 3\) with compact connected boundary \(N\). Suppose that

1. \(R_g \geq -n(n-1)\) in a neighborhood of \(N\).
2. \(N\) has mean curvature \(H_N \leq n - 1\).
3. \(N\) is locally weakly outermost.
4. \(N\) does not carry a metric of positive scalar curvature.

Then \(\exists\) a nbd \(U\) of \(N\) that \((U, g|_U)\) is isometric to \(([0, \infty) \times N, dt^2 + e^{2t}g|_N)\) where \((N, g|_N)\) is Ricci flat.
Conditions 3 and 4 are needed

The following examples are not isometric to the resulting warped product.

1. Toroidal Kottler metric with $m > 0$: $[r_0, \infty) \times T^{n-1}$ with the metric

$$g = \left( r^2 - \frac{2m}{r^{n-2}} \right)^{-1} dr^2 + r^2 h$$

where $r_0 = (2m)^{\frac{1}{n}}$ and $h$ is a flat metric on $T^{n-1}$.

$\Rightarrow R_g = -n(n-1)$ and $N \cong T^{n-1}$ does not carry a metric of positive scalar curvature. But $N$ is not weakly outermost.

2. AdS Schwarzschild manifold: for $m > 0$, $[r_m, \infty) \times S^{n-1}$ with the metric

$$g = \left( 1 + r^2 - \frac{2m}{r^{n-2}} \right)^{-1} dr^2 + r^2 g_{S^{n-1}}, \quad r_m = (2m)^{\frac{1}{n-2}}.$$

$\Rightarrow R_g = -n(n-1)$ and $N$ is weakly outermost. But $N$ carries a metric of positive scalar curvature.
Theorem (Galloway and J- ’19)

\((M^n, g), n \geq 3\) : complete, noncompact with compact connected boundary \(N\).
Suppose that

1. \(R_g \geq -n(n - 1)\) in a neighborhood of \(N\).
2. \(N\) has mean curvature \(H_N \leq n - 1\).
3. \(N\) is locally weakly outermost.
4. \(N\) does not carry a metric of positive scalar curvature.
5. \(\exists\) a nonzero function \(f\) satisfying \(\text{Hess}_g f - fg = 0\).

Then \((M, g)\) is isometric to \(([0, \infty) \times N, dt^2 + e^{2t} g|_N)\) where \((N, g|_N)\) is Ricci flat.

Using the local version, one can prove such function \(f\) must be constant on \(N\).
⇒ The theorem follows by the proposition.
Remark

One can obtain a similar characterization of the shrinking end by using \((M, g, g)\).

**Theorem (Galloway and J- ’19)**

\((M^n, g), n \geq 3 : complete, noncompact with compact connected boundary \(N\).
Suppose that

1. \(R_g \geq -n(n - 1)\) in a neighborhood of \(N\).
2. \(N\) has mean curvature \(H_N \leq -(n - 1)\).
3. \(N\) is locally weakly outermost (the inequality changes to \(H_\Sigma < -(n - 1)\)).
4. \(N\) does not carry a metric of positive scalar curvature.
5. \(\exists\) a nontrivial solution \(f\) of \(\text{Hess}_g f - fg = 0\).

Then \((M, g)\) is isometric to \([0, \infty) \times N, dt^2 + e^{-2t}g|_N\) where \((N, g|_N)\) is Ricci flat.
Thank you for your attention.